Global Jacquet-Langlands

§ Adeles

Ref: math.stanford.edu/~conrad/papers/adelictop.pdf

k global field examples: $\mathbb{Q}$, $\mathbb{F}_q(x)$ fin. extn.

For each place v of k, get local field k_v

valuation $|\cdot|_v$

$$\mathbb{A}_k := \prod_{\substack{v \\ \text{place of } k}}{}' k_v = \{ (x_v)_v \mid x_v \in k_v, \text{ for a.a. } v, x_v \in \mathcal{O}_{k_v} \}$$

$$\mathbb{A}_k^\times \xrightarrow{\text{inherit topology}} SL_2(\mathbb{A}_k) \subset M_2(\mathbb{A}_k)$$

$$x \mapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$$

Topology

k_v topology from $|\cdot|_v$ $\qquad V_v \subseteq k_v$ open

$\mathbb{A}_k$ open basis $\prod_{v \in S} V_v \times \prod_{v \notin S} \mathcal{O}_{k_v}$ $\qquad S$ fin. subset of places containing all infinite places.

$\mathbb{A}_k^\times$ — — $\prod_{v \in S} V_v \times \prod_{v \notin S} \mathcal{O}_{k_v}^\times$ $\qquad V_v \subseteq k_v^\times$ open

$k \hookrightarrow \mathbb{A}_k$ image is discrete

$x \mapsto (x, x, \cdots)$

$\mathbb{A}_k / k$ compact

$k = \mathbb{Q}$ $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_{p}' \mathbb{Q}_p \hookleftarrow \mathbb{Q}$

p prime number

$$\left(\mathbb{R} \times \prod_p \mathbb{Z}_p \right) \cap \mathbb{Q} = \mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$$

$$\left(\underbrace{\left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \prod_p \mathbb{Z}_p}_{\text{open}} \right) \cap \mathbb{Q} = \{0\}$$

$$\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \times \prod_p \mathbb{Z}_p \right) + \mathbb{Q} = \mathbb{A}_{\mathbb{Q}} \Rightarrow \mathbb{A}_{\mathbb{Q}} / \mathbb{Q} \simeq \mathbb{R}/\mathbb{Z} \times \prod_p \mathbb{Z}_p$$

Normalise $|\cdot|_v$

uniformiser

$$|\varpi_v|_v = q_v^{-1} \qquad q_v := \#|\mathcal{O}_{k_v}/\mathfrak{p}_v| \qquad v \text{ finite.}$$

$$(\varpi_v) = \mathfrak{p}_v.$$

$|x|_v$ = usual abs. val. of $x \in \mathbb{R}$ $\qquad v$ real

$|x|_v$ = square of usual abs. val. of $x \in \mathbb{C}$ $\qquad v$ complex

Ideles $\quad \mathbb{A}_k^\times = \prod_v{}' k_v^\times \quad$ w.r.t. $\quad \mathcal{O}_v^\times$.

$$|x|_{\mathbb{A}_k} = \prod_v |x_v|_v \qquad x \in \mathbb{A}_k^\times \qquad x_v \in \mathcal{O}_v^\times \text{ for a.a. } v.$$

$$|x|_{\mathbb{A}_k} = 1 \quad \text{if } x \in k^\times \qquad (\text{product formula})$$

$$|\ |_{\mathbb{A}_k} : k^\times \backslash \mathbb{A}_k^\times \longrightarrow \mathbb{C}^\times.$$

§ Some groups

$G = GL_2 / k$ algebraic group $/ k$.

$G(k)$ $G(k_v)$ topology inherited from $G(k_v) \subset M_2(k_v) \simeq k_v^4$

v fin. $G(\mathcal{O}_k) \subset G(k_v)$ max cpt. subgp

v real $O(2) \subset G(\mathbb{R})$

v complex $U(2) \subset G(\mathbb{C})$

$\Big\}$ K_v

$$G(\mathbb{A}) := \prod_v{}' G(k_v) \quad \text{w.r.t.} \quad \{K_v\}_v.$$

$$K := \prod_v K_v$$

$G \supset B = UH$ $\qquad$ $G \supset Z$ centre of G.

Borel $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\qquad$ unipotent radical $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ $\qquad$ $\begin{pmatrix} t & \\ & t \end{pmatrix}$

Iwasawa decomposition :

$$G(k_v) = U(k_v)\, H(k_v)\, K_v .$$

$$G(\mathbb{A}) = U(\mathbb{A})\, H(\mathbb{A})\, K$$

Siegel set

approximate fund. domain

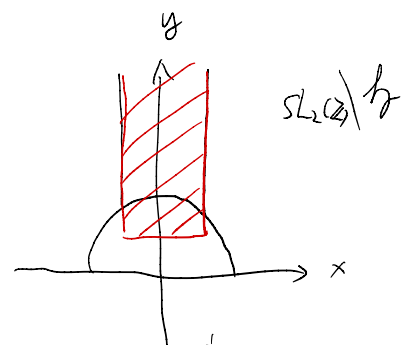
$$\mathfrak{S} = \underset{\substack{\cap \\ U(\mathbb{A})}}{\Omega_U}\ \underset{\substack{\cap \\ H(\mathbb{A})}}{\Omega_H}\ \underset{\text{rel. cpt.}}{H_\infty(c)}\ K$$

$$H_\infty(c) \subset H(\mathbb{A}_\infty) \qquad c > 0$$

$$H_\infty(c) = \left\{ \begin{pmatrix} \underline{t_1} & \\ & \underline{t_2} \end{pmatrix} \;\middle|\; \begin{array}{ll} \underline{t_1} = (t_1, \cdots, t_1) & t_1 \in \mathbb{R}_{>0} \\ \underline{t_2} = (t_2, \cdots, t_2) & t_2 \in \mathbb{R}_{>0} \\ t_1 t_2^{-1} > c & \end{array} \right\}$$

For appropriate Ω_U, Ω_H, c small

$$G(\mathbb{A}_k) = G(k)\, \mathfrak{S} .$$



Ex :

$$SL_2(\mathbb{R})/SO(2) \simeq \mathfrak{h}$$

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} \longleftarrow x + iy$$

$$g \longmapsto g.i = \frac{ai+b}{ci+d}$$

$$\mathfrak{S} = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} SO(2) \;\middle|\; \begin{array}{l} x \in [-\frac{1}{2}, \frac{1}{2}] \\ y > c \end{array} \right\}$$

§ cuspidal functions

Ref: For $\mathcal{A}$ not L^2 Moeglin-Waldspurger

Fix a Haar measure on $G(\mathbb{A})$

$$L^2(G(k)\backslash G(\mathbb{A})) = \left\{ \varphi \in \mathcal{F}(G(\mathbb{A}), \mathbb{C}) \;\middle|\; \begin{array}{l} \varphi(\gamma g) = \varphi(g) \quad \forall \gamma \in G(k) \\ \int_{G(k)\backslash G(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty \end{array} \right\}$$

$\circlearrowleft$ $G(\mathbb{A}) \ni g$ $\quad \varphi(\cdot) \mapsto \varphi(\cdot g)$

$\circlearrowleft_{Z(k)\backslash Z(\mathbb{A})}$

Decompose according to characters of $Z(k)\backslash Z(\mathbb{A}) \simeq k^\times \backslash \mathbb{A}^\times \quad (\sim \mathbb{R}^\times_{>0})$

ω a character of $Z(k)\backslash Z(\mathbb{A})$

$$L^2(G(k)\backslash G(\mathbb{A}), \omega) = \left\{ \varphi \in \mathcal{F}(G(\mathbb{A}), \mathbb{C}) \;\middle|\; \begin{array}{l} \varphi(\gamma g) = \varphi(g) \quad \forall \gamma \in G(k) \\ \varphi(zg) = \omega(z)\varphi(g) \quad \forall z \in Z(\mathbb{A}) \\ \int_{G(k)Z(\mathbb{A})\backslash G(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty \end{array} \right\}$$

Aside:

$\varphi = \int \varphi_\omega \, d\omega$ $\qquad f_g(z) := \varphi(zg)$

$(Z(k)\backslash Z(\mathbb{A}))^\wedge$ = set of characters of $Z(k)\backslash Z(\mathbb{A})$

$f_g(z) = \int f_{g,\omega} \, \omega(z) \, d\omega$

$\varphi(g) = \int f_{g,\omega} \, d\omega$